Pergamon

PII: S0021-8928(96)00105-0

J. Appl. Maths Mechs, Vol. 60, No. 5, pp. 845–850, 1996

0021-8928/96 $24.00+0.00

PERIODIC AND STOCHASTIC SELF-EXCITED OSCILLATIONS IN A SYSTEM WITH HEREDITARY-TYPE DRY FRICTION†

V. S. METRIKIN, R. F. NAGAYEV and V. V. STEPANOVA

Nizhnii Novgorod, St Petersburg

(*Received* 6 *October* 1994)

The self-excited oscillations of an oscillator which is coupled by dry friction to a base moving at a constant velocity (Fig. 1) is considered. It is assumed that the coefficient of sliding friction f^* is constant and that the coefficient of static friction is a piecewise-linear function of the duration t_k of the preceding interval of prolonged contact between the body and the base (Fig. 2) [1]. A classification of the simplest periodic and steady-state stochastic self-excited oscillations of the oscillator is given and the domains of their existence in the parameter space of the system are constructed. The domains of transient-type motion, within which periodic modes of arbitrary complexity exist, are analysed in detail. In particular, the equations of the so-called inaccessible boundaries [2] are constructed in explicit form. A denumerable set of different periodic trajectories of the dynamical system under consideration exists in a small neighbourhood of these boundaries.

In [3], attention was drawn for the first time to the possibility of exciting stochastic self-excited oscillations. The case of a smooth, monotonically increasing characteristic of the coefficient of static friction of the exponential type was subsequently investigated in detail [4]. Here, the occurrence of finite domains in the parameter space of the problem where transient-type motions exist was not detected.

1. EQUATIONS OF MOTION. PHASE SPACE. THE SUCCESSION FUNCTION

The equations of motion of the essentially non-linear system under consideration within the intervals in which slipping and prolonged contact occurs have the form

$$m\ddot{x}+cx+f_*P\operatorname{sign}(\dot{x}-V)=0,\quad \dot{x}\neq V \tag{1.1}$$

$$\dot{x}=V,\quad c|x|\leqslant fP \tag{1.2}$$

where m is the mass, P is the weight of the body, c is the stiffness of the spring and, in accordance with what has been said above, the coefficient of static friction is equal to

$$f=\begin{cases} f_*+(f^*-f_*)t_k/t_*, & 0<t_k<t^* \\ f^*, & t_k>t^* \end{cases} \tag{1.3}$$

We assume that, when $-t_k<t<0$, the body is in contact with the belt and, when $t=0$, we have $x=f(t_k)P/c$, $\dot{x}=V$. Later, the motion occurs with a lag ($\dot{x}<V$) and is completed when $t=t^{(1)}$, when $\dot{x}=V$ ($x=x^{(1)}$) for the first time. If, in this case, the acceleration $\ddot{x}$, when $t=t^{(1)}+0$ and $\dot{x}>V$ by virtue of Eq. (1.1), is positive ($cx^{(1)}+f_*P<0$), then there is an instantaneous change in the direction of the slipping and an interval of motion with a lead ($\dot{x}>V$) begins. In the opposite case when $t>t^{(1)}$, a subsequent prolonged contact is inevitable.

A preliminary analysis [4] shows that, generally speaking, only a finite number of subsequent intervals with lags and leads is possible and that this number is greater, the greater the difference $f(t_k)-f_*$. In this case, the successive intervals t_k and t_{k+1} of prolonged contact are related to one another by a point mapping T of the half-line $L(\eta)$ into itself

$$\Psi(\eta_{k+1})=\varphi(\eta_k) \tag{1.4}$$

where

†*Prikl. Mat. Mekh.* Vol. 60, No. 5, pp. 859–864, 1996.

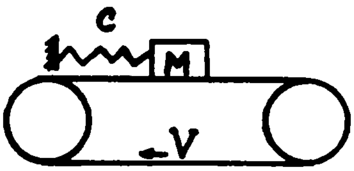

Fig. 1.

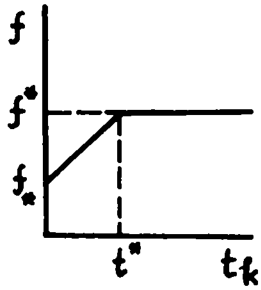

Fig. 2.

$$
\begin{aligned}
&\Psi(\eta) = \beta\eta - \varepsilon(\eta)\\
&\varphi(\eta) = 1 - (-1)^j[\varepsilon(\eta) - 2j + 1]\\
&2(j-1) < \varepsilon(\eta) < 2j \quad (j = 1, 2, 3, \ldots)\\
&\varepsilon(\eta) = \begin{cases}\eta, & 0 < \eta < \varepsilon_*\\ \varepsilon_*, & \eta > \varepsilon_*\end{cases}\\
&\beta = \frac{\sqrt{mc}\,V}{f_* P}, \quad \varepsilon_* = \frac{f^* - f_*}{f_*}, \quad \eta_k = \frac{\varepsilon_* t_k}{t^*}
\end{aligned}
\tag{1.5}
$$

and the value of j is equal to the number of intervals where slipping occurs when $t_k < t < t_{k+1}$. It should be noted that the ith slipping stage exists if $\varepsilon(\eta_k) > 2(i-1)$. The qualitative forms of the functions Ψ and φ for $2 < \varepsilon_* < 4$ are shown in Fig. 3 by the solid and dashed lines respectively.

2. THE SUCCESSION FUNCTIONS

2.1. It follows from (1.4), (1.5) and Fig. 3 that, when $\beta > 2$, there is always one stable fixed point $\eta_0^* = 0$ of the mapping T which corresponds to harmonic oscillations of the body

$$x = \frac{f_* P}{c} + V\sqrt{\frac{m}{c}}\sin\sqrt{\frac{c}{m}}\,t$$

without zones of prolonged contact with the belt. The line $\beta = 2$ in the plane of the parameters β, ε_* is the boundary between the trajectories of the dynamical system (TDS) which correspond to uninterrupted motions and periodic or stochastic types of motion with prolonged contact of the body. As β $(0 < \beta < 2)$ decreases, the number of simple fixed points of the mapping T, which are determined from the equation $\Psi(\eta^*) = \varphi(\eta^*)$, increases. In this case, the fixed point η_0^* becomes unstable and simple stable fixed points η_i^* exist only in the domain of the parameters $\eta_i^* > \varepsilon_*$ and are determined from the relations

$$\eta_i^* = \beta^{-1}[1 + \varepsilon_* - (-1)^i(\varepsilon_* - 2i + 1)] \tag{2.1}$$

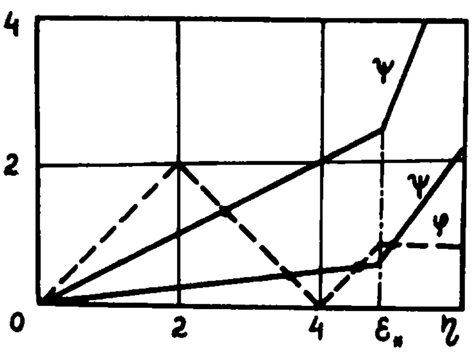

Fig. 3.

and the $i/2$-cycle trajectories of the dynamical system with a valued of prolonged contact η_i^* of the body with the belt [4] which correspond to them.

The remaining simple fixed points η_k^{**} belonging to the intervals $0 < \eta_k^{**} < \varepsilon_*$ are unstable and are defined in terms of the parameters of the system by the following formula

$$\eta_k^{**}[1+(-1)^{k+1}(1-2k)]/[\beta-1+(-1)^k] \quad (k=2,3\ldots) \tag{2.2}$$

It also follows from (1.4) and (1.5) that, if $2 < \varepsilon_* < 4$ ($j = 2$) then: when $0 < \beta < 4/\varepsilon_*$, a stable 1-cycle periodic TDS with a time η_2^* of prolonged contact between the body and the belt exists; stable cycles of multiple points of the mapping T^n exist when $4/\varepsilon_* < \beta < (2 + \varepsilon_*)/\varepsilon_*$, and 1-cycle stochastic TDS occur when $(2 + \varepsilon_*)/\varepsilon_* < \beta < 2$.

If $4 < \varepsilon_* < 6$ ($j = 3$), then, similarly, we have that 3/2-cycle stable periodic TDS exist when $0 < \beta < 2 - 4/\varepsilon_*$ with a time η_3^* of prolonged contact between the body and the belt, cycles of multiple points of the mapping T^n are observed when $2 - 4/\varepsilon_* < \beta < (2 + \varepsilon_*)/\varepsilon_*$ and, finally, when $(2 + \varepsilon_*)/\varepsilon_* < \beta < 2$, 1-cycle stochastic trajectories of the dynamic system with prolonged contacts occur.

Continuing to consider the intervals of change, the following can be shown in the general case.

1. Trajectories of the dynamical system without prolonged contacts of the body occur in the domain G_0 ($\beta > 2$, $\varepsilon_* \geq 0$).

2. Stable periodic $j/2$-cycle trajectories of the dynamical system with a time η_j^* of prolonged contact occur in the domains $G_{j/2}^p$($2(j - 1) < \varepsilon_* < 2j$; $0 < \beta < 2j/\varepsilon_*$ for even j and $0 < \beta < 2 - (2(j - 1)/\varepsilon_*$ for odd j.

3. Cycles of multiple points exist in the domains G_j^k of the plane of the parameters β, ε_*, and the boundaries of these domains are given by the relations

$$\beta_s = 1 + 2/\varepsilon_* \tag{2.3}$$

$$\beta_- = \begin{cases} 2j/\varepsilon_*, & j = 2,4,6,\ldots \\ 2 - 2(j-1)/\varepsilon_*, & j = 3,5,7,\ldots \end{cases}$$

4. $j/2$-cycle stochastic trajectories of the dynamical system occur in the domains $G_{j/2}^s$ between the boundaries

$$\beta = 1 + 2/\varepsilon_*, \quad \beta = 1 + 1/(j-1) \quad (j = 2,4,6,\ldots) \tag{2.4}$$

The subdivision of the plane of the parameters β, ε_* into domains of existence of the trajectories of the dynamical system indicated above is shown in Fig. 4.

3. COMPLEX TYPES OF TRAJECTORY OF THE DYNAMICAL SYSTEM

3.1. Suppose that $2 < \varepsilon_* < 4$. It follows from the results presented above that, as β increases from zero, the 1-cycle stable periodic trajectories of the dynamical system become unstable when $\varepsilon_*^0 = 4/\alpha$ ($\alpha = \beta - 1$). It can be shown that, in this case, a pair of stable fixed points of the mapping T^2 is generated which correspond to stable periodic trajectories of the dynamical system with a doubled period which, when $\varepsilon_*^1 = 4(1 + \alpha^2)$ lose stability and a stable fourfold periodic trajectory of the dynamical system is generated (stable fixed points of the mapping T^4 appear) which, when $\varepsilon_*^2 = 4(1 - \alpha^2 + \alpha^3)/(1 + \alpha^4)$ loses stability, and so on.

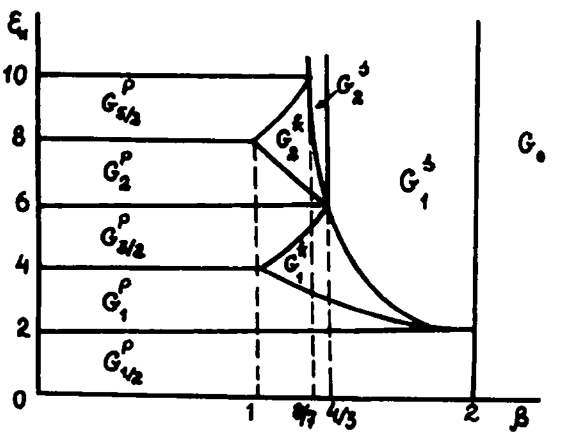

Fig. 4.

On continuing to vary ε_*, we can obtain that a doubling of the multiplicity of the period of the periodic trajectories of the dynamical system occurs on crossing the boundary Γ_φ^n and that the equations of these trajectories can be described by the formula

$$\varepsilon_*^n = \frac{4}{1+\alpha^{2^n}} \sum_{p=0}^{s} \alpha^{(2^{2p}-1)} \prod_{k=2p+1}^{n-1} (1-\alpha^{2^k}) \quad (3.1)$$

$s = n/2$ for even n and $s = (n-1)/2$ for odd n, where ε_*^n defines 2^n- and 2^{n+1}-fold periodic trajectories of the dynamical system ($n = 0, 1, 2, \ldots$).

When $\beta > \beta_-$, there is no mapping T of the simple stable fixed points, but as has been shown above, there are stable points of the mapping T^n. In this case, the Lyapunov index [5] is negative ($\lambda < 0$). When $\beta > \beta_s$, the Lyapunov index changes sign and becomes positive ($\lambda > 0$) which corresponds to the onset of chaos.

The corresponding form of the succession functions for the mappings T and T^2 and the two sets of values for the parameters $\varepsilon_* = 3$, $\beta = 1.4$; $\varepsilon_* = 3$, $\beta = 1.75$ is shown in Fig. 5.

In the domains (2.3) it is possible to separate out the inaccessible boundaries Γ_-^n; periodic trajectories of the dynamical system of as high a multiplicity as may be desired exist in as small a neighbourhood of these boundaries as may be desired, and the equations of the boundaries of the domains of existence of these trajectories can be written in the form

$$\varepsilon_* = 4\frac{\alpha^n - 1}{\alpha^{n+1} - 1}, \quad \varepsilon_* = \frac{4}{1+\alpha^{n+1}} \quad (3.2)$$

and the equations of the boundaries Γ_-^n in the form

$$\varepsilon_* = 4\frac{1+\alpha-\alpha^{n+1}}{1+\alpha} \quad (3.3)$$

The domains of existence of complex trajectories of the dynamical system are shown in the plane of the parameters α, ε_* in Fig. 6. The inaccessible boundaries are denoted by a dotted and dashed line. Bifurcations of the doubling of the period are observed when the parameters are changed to values lying outside the hatched regions on the side of the inaccessible boundaries. Complex trajectories of the dynamical system, including stochastic trajectories, exist in the hatched regions.

3.2. Suppose that $4 < \varepsilon_* < 6$. In this case, the inaccessible boundaries are defined for different n by the formula

$$\varepsilon_* = 4[1+\alpha^{n+1}/(1+\alpha)] \quad (3.4)$$

They are denoted, in the plane of the parameters α, ε_* in Fig. 7, by dot–dash curves. Cycles of $(n+1)$-fold points of a mapping T^{n+1} exist for values of the parameters α, ε_* from domains with boundaries which are defined by the relations

$$\varepsilon_* = 4\frac{\alpha^n + 1}{\alpha^{n+1} + 1}, \quad \varepsilon_* = \frac{4}{1-\alpha^{n+1}} \quad (3.5)$$

Here, as in the case when $2 < \varepsilon_* < 4$, the inaccessible boundaries bunch together as n increases, approaching one another and tending to the straight line $\varepsilon_* = 4$. In this case, the Feigenbaum process of doubling of the period is again observed.

The straight line $\varepsilon_* = 4(\alpha + 1)$ can be singled out in the plane of the parameters α, ε_* in Fig. 7. Domains G_3^p where three-fold periodic trajectories of the dynamical system exist lie on both sides of this line and they are bounded

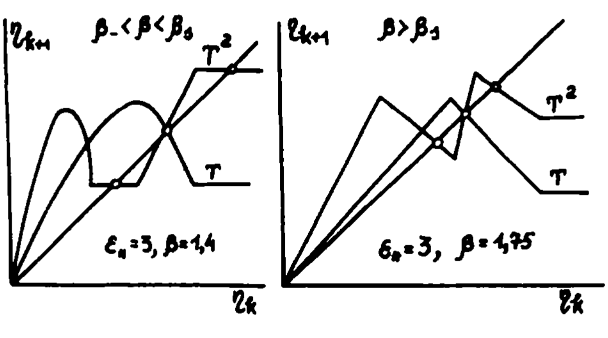

Fig. 5.

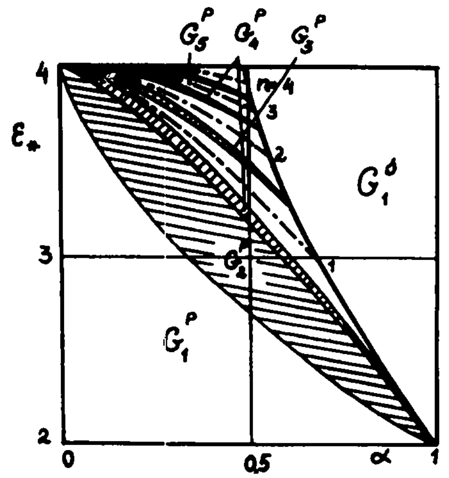

Fig. 6.

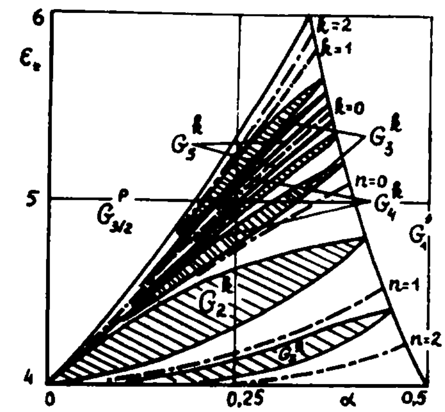

Fig. 7.

by the curves

$$\varepsilon_* = 4\frac{1+\alpha+\alpha^3}{1+\alpha^3}, \quad \varepsilon_* = 4\frac{1+\alpha}{1-\alpha^3} \tag{3.6}$$

$$\varepsilon_* = 4\frac{1+\alpha}{1+\alpha^3}, \quad \varepsilon_* = \frac{1+\alpha-\alpha^2}{1-\alpha^3} \tag{3.7}$$

between which, in turn, domains of four-fold, five-fold and so on periodic trajectories of the dynamical system exist, bounded by the curves

$$\varepsilon_* = 4\frac{1+\alpha+\alpha^n}{1+\alpha^{n+1}}, \quad \varepsilon_* = 4\frac{1+\alpha}{1-\alpha^{n+1}} \quad (n = 2,3,4,\ldots) \tag{3.8}$$

Here, the process of the doubling of the period is also observed as n increases as the straight line is approached.

It can be shown that there is an even number of inaccessible boundaries in the plane of the parameters α, ε_*

$$\varepsilon_{*k} = 4(1-\alpha^{k+2})/(1-\alpha) \tag{3.9}$$

These inaccessible boundaries separate pairs (with the same number of n-fold cycles) of domains where periodic trajectories of the dynamical system exist.

For instance, the curve $\varepsilon_{*1} = 4(1 + \alpha + \alpha^2)$ separates two domains of four-fold periodic trajectories of the dynamical system. On the two sides of the curve $\varepsilon_{*2} = 4(1 + \alpha + \alpha^2 + \alpha^3)$ there are two domains of five-fold trajectories of the dynamical system, and, between them, there are domains of six-fold, seven-fold and so on trajectories of the dynamical system. This process of the occurrence of pairs of domains of complex periodic trajectories of the dynamical system continues in the direction of change to the domain of 3/2-cycle periodic trajectories of the dynamical system. The nearer curve (3.9) is to this domain, the higher the multiplicity of the simplest trajectory of the dynamical system and the kth inaccessible boundary separates the domains of the $(k + 3)$-, $(k + 4)$- and $(k + 5)$-fold periodic trajectories of the dynamical system, the equations of which are written, in the general case, in the following manner

$$\begin{aligned} \varepsilon_* &= [4(1-\alpha^{k+2})/(1-\alpha)+\alpha^{n+k}]/(1+\alpha^{n+k+1}) \\ \varepsilon_* &= 4(1-\alpha^{k+2})/(1-\alpha)(1-\alpha^{n+k+1}) \end{aligned} \tag{3.10}$$

and, in the case of the domains located above the latter domains, as

$$\begin{aligned} \varepsilon_* &= 4(1-\alpha^{k+2})/(1-\alpha)(1+\alpha^{n+k+1}) \\ \varepsilon_* &= [4(1-\alpha^{k+2})/(1-\alpha)-\alpha^{n+k}]/(1-\alpha^{n+k+1}) \end{aligned} \tag{3.11}$$

In (3.10), and (3.11) $n = 2, 3, 4, \ldots$; $k = 0, 1, 2, \ldots$; the multiplicity of the periodic trajectories of the dynamical system is denoted by $n + k + 1$.

The domains of complex trajectories of the dynamical system, including stochastic trajectories, are shown in Fig. 7.

By varying ε_* in a similar manner, it can be shown that the situation described above, involving the occurrence of pairs of domains of existence of trajectories of the dynamical system which may be as complex as desired in the neighbourhood of the inaccessible boundaries, holds in the whole plane of the parameters of the system.

REFERENCES

1. ISHLINSKII A. Yu. and KRAGEL'SKII I. V., On discontinuities accompanying friction. *Zh. Tekhn. Fiz.* **14**, 4/5, 276–282, 1944.
2. BUTENINA N. N. and METRIKIN V. S., On the inaccessible boundaries of a point mapping of a circle into itself. In *Differential and Integral Equations*. Gor'k. Gos. Univ., Gor'kii, 80–84, 1981.
3. KASHCHENEVSKII L. Ya., Stochastic self-excited oscillations accompanying dry friction. *Inzh.-Fiz. Zh.* **47**, 1, 143–147, 1984.
4. VETYUKOV M. M., DOBROSLAVSKII S. V. and NAGAYEV R. F., Self-excited oscillations in a system with a characteristic of hereditary-type dry friction. *Izv. Akad. Nauk SSSR. MTT* 1, 23–28, 1990.
5. SCHUSTER H., *Deterministic Chaos*. Mir, Moscow, 1988.

Translated by E.L.S.